HW-1 I

1 Gaussian integral over $\tilde{\phi}$:→

$$\int \mathcal{D}\tilde{\phi}\, \Psi_{IR}\, \Psi_{UV} = \int \mathcal{D}M \int \mathcal{D}\tilde{\phi}\; e^{-S_0[M]}\; e^{\int \frac{d^dx}{\ell^d}\left(\tilde{\phi}\,\mathcal{O} - \frac{1}{2h}\left(\tilde{\phi}+g\right)^2\right)}$$

e.o.m.

$$\mathcal{O} = \frac{1}{h}\left(\tilde{\phi}+g\right)$$

$$\therefore\ \tilde{\phi} = h\mathcal{O} - g.$$

$$\therefore\ \tilde{\phi}\,\mathcal{O} - \frac{1}{2h}\left(\tilde{\phi}+g\right)^2 = \left(h\mathcal{O}-g\right)\mathcal{O} - \frac{1}{2h}\left(h^2\mathcal{O}^2\right)$$

$$= \frac{1}{2}h\mathcal{O}^2 - g\mathcal{O}.$$

$$e^{-S} = \int \mathcal{D}M\, e^{-S_0}\, e^{-\int \frac{d^dx}{\ell^d}\left(g\mathcal{O} - \frac{1}{2}h\mathcal{O}^2\right)}.$$

thus: $S = S_0[M] + \int \frac{d^dx}{\ell^d}\left(g\mathcal{O} - \frac{1}{2}h\mathcal{O}^2\right).$

2a $$\frac{\delta S}{\delta \mathcal{O}} = \frac{1}{\ell^d}\left(g - h\mathcal{O}\right)$$

2b. $$\partial_\ell S = \frac{1}{\ell^d}\left\{-\frac{d}{\ell}g\mathcal{O} + \frac{dh}{2\ell}\mathcal{O}^2 + \left(\partial_\ell g\right)\mathcal{O} - \frac{1}{2}\left(\partial_\ell h\right)\mathcal{O}^2\right\}.$$

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3. $$\mathcal{H}\left(\tilde{\phi},\tilde{\pi}\right) = \frac{1}{2\ell}\left(\ell^d\tilde{\pi}^2 + \frac{1}{\ell^d}m^2\tilde{\phi}^2\right)$$

~~$e^{\int \tilde{\phi}\,\mathcal{O}\frac{d^dx}{\ell^d}}$ $\tilde{\phi}$ was the enters as the source of $\mathcal{O}$.~~

~~"Legendre transform".~~

~~→ $\tilde{\phi}$ can be replaced by : ?~~

S

The effective action is defined as

$$e^{-S} = \int \mathcal{D}\tilde{\phi} \left\langle e^{\int \tilde{\phi}\mathcal{O} \frac{d^dx}{l^d}} \cdots \right\rangle$$

In the Hamiltonian then $\tilde{\phi}$ can be replaced by $-l^d \frac{\delta S}{\delta \mathcal{O}}$.

And $\tilde{\pi} = -i \frac{\delta}{\delta \tilde{\phi}}$ is then: $\frac{i}{l^d}\mathcal{O}$.

Now the

$$\mathcal{H}\left(-l^d \frac{\delta S}{\delta \mathcal{O}}, \frac{i}{l^d}\mathcal{O}\right) = \frac{1}{2l}\left\{-\frac{1}{l^d}\mathcal{O}^2 + \frac{m^2}{l^d}(g - h\mathcal{O})^2\right\}$$

(4) $$\partial_l S = \mathcal{H}\left(\frac{i}{l^d}\mathcal{O}, -l^d \frac{\delta S}{\delta \mathcal{O}}\right)$$ (Hamilton-Jacobi Equation)

$$\Rightarrow \frac{1}{l^d}\left\{-\frac{d}{l} g\mathcal{O} + \frac{dh}{2l}\mathcal{O}^2 + (\partial_l g)\mathcal{O} - \frac{1}{2}(\partial_l h)\mathcal{O}^2\right\}$$

$$= \frac{1}{2l}\frac{1}{l^d}\left\{-\mathcal{O}^2 + m^2 g^2 - m^2 h^2 \mathcal{O}^2 - 2m^2 g h \mathcal{O}\right\}.$$

Comparing coefficients of $\mathcal{O}^2$:

$$\frac{dh}{\cancel{2l}} - \frac{1}{\cancel{2}}\partial_l h = -\frac{1}{\cancel{2l}} - \frac{m^2h^2}{\cancel{2l}}$$

$$+ l\partial_l h = dh + 1 - m^2 h^2$$

$$h_* = \frac{1}{2m^2}(d \pm 2\nu)$$

$$\nu = \sqrt{\frac{d^2}{4} + m^2}$$

$$\cancel{\frac{-d \pm \sqrt{\cdots}}{-2m^2} = \frac{1}{2m^2}\left(d \mp \sqrt{\frac{d^2}{4}+m^2}\right)}$$

coeff of $\mathcal{O}$:

$$-\frac{d}{l}g + \partial_l g = -\frac{m^2 g h}{l}$$

$$l\partial_l g = -m^2 g h + dg$$

$$l\partial_l g = -m^2 g \frac{1}{2m^2}(d \pm 2\nu) + dg$$

$$= -\frac{1}{2} g d \mp \nu g + dg$$

$$= g\left(\frac{d}{2} \mp \nu\right)$$

$$g \sim l^{\frac{d}{2} \mp \nu} \Rightarrow [\mathcal{O}] \Rightarrow \Delta_\pm \checkmark$$

Part of solution (II)

For the Euclidean Metrics of a Black hole

$$ds^2 = f\,d\tau^2 + \frac{dr^2}{f}$$

Let:

$f(r_+) = 0$ so; $r = r_+$ is the location of the black hole horizon.

thus $f(r) \sim f'(r_+)\,(r - r_+)$.

Near $r = r_+$ the metric looks like:

$$ds^2 \sim f'(r_+)\,(r - r_+)\,d\tau^2 + \frac{dr^2}{f'(r_+)(r - r_+)}$$

define: $$d\rho = \frac{dr}{\sqrt{f'(r_+)(r - r_+)}}$$

thus:

$$\rho = 2\sqrt{\frac{r - r_+}{f'(r_+)}} \Rightarrow (r - r_+) = f'(r_+)\,\frac{\rho^2}{4}.$$

$\therefore$ The near horizon metric thus is:

$$ds^2 \sim f'(r_+)\,f'(r_+)\,\frac{\rho^2}{4}\,d\tau^2 + d\rho^2$$

$$= \rho^2 \left(\frac{f'(r_+)^2}{4}\right) d\tau^2 + d\rho^2$$

Writing $\frac{f'(r_+)}{2}\tau = \theta$, then this metric is just flat 2-dim Euclidean metric written in polar coordinate provided θ has periodicity $0 < \theta < 2\pi$.

This implies: τ has periodicity $\beta = \frac{2\pi}{\frac{f'(r_+)}{2}} = \frac{4\pi}{f'(r_+)}$.

Thus the inverse temperature of the BH is $\boxed{\beta = \frac{4\pi}{f'(r_+)}}$